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SOME
GEOMETRICAL PORISMS,
WITH
EXAMPLES
OF THEIR
APPLICATION TO THE SOLUTION OF PROBLEMS.

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THE nature of those mathematical propositions, which were called Porisms by the ancient geometers, is now no longer a matter of uncertainty. The relation which they bear to other mathematical truths, the way in which they may at first have been observed, the kind of analysis to be employed in their investigation, their application to the solution of problems, have all been considered by some eminent mathematicians of the present age.

These propositions appear to have been held in high estimation by the mathematicians of antiquity, because of their great use in the analysis of difficult problems, as we learn from the writings of PAPPUS of Alexandria: And some specimens, which late inquirers into this subject have given us, of their application to the solution of problems, seem to justify his very high character of them.

The following paper contains some porisms intimately connected with each other, and which seem capable of being applied to the solution of a number of geometrical problems. Examples of their application are added, some of which are pro-

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blems that have been long known, and others are new; but the constructions of the former, it is believed, differ from any hitherto published. Although there are several of these examples, in appearance, little related to each other, yet their solutions are affected by the same general principle, which is also the foundation of all the porisms.

PROP. I. PORISM, Fig. 4, 5. Pl. I.

Let AB , AC , be two straight lines given by position, let B , C , be given points in these lines, a point H may be found, such, that any circle whatsoever passing through A , the intersection of the given lines, and H the point which may be found, shall cut off from the given lines segments BD , CE , adjacent to the given points, and having to each other the given ratio of α to β .

Suppose the porism to be true, and that the point H is found. If a circle be described through H , A , and B one of the given points, it must also pass through C the other given point, that the proposition may be universally true. Therefore H is in the circumference of a given circle. Join BH , CH , DH , EH . The angle DHE is equal to DAE , that is, to BHC , (fig. 4.) or DHE is the supplement of DAE , (fig. 5.) and therefore equal to BHC ; hence BHD is equal to CHE , but BDH is equal to CEH , therefore the triangles BDH , CEH , are equiangular, and BH is to HC as BD to EC , that is, by hypothesis, in the given ratio of α to β ; therefore if BC be joined, the triangle BHC is given in species, and BC being given, BH and HC are given; therefore the point H is given, which was to be found.

If the segments BD , CE , cut off from the given lines, lie in the same direction with respect to AB , AC , (fig. 4.) the point H will be in the same segment of the circle with the angle BAC ; but if BD , CE , lie in contrary directions to AB , AC , (fig. 5.)

then H will be in that segment of the circle upon which BAC stands.

The point H will be found by the following construction: Describe a circle through the points A, B, C. Join BC, which divide at G, so that BG may be to GC in the given ratio of BD to CE, that is of α to β , and if the segments to be cut off are to lie in the same direction with AB, AC, find F the vertex of the segment upon which the angle BAC stands, (fig. 4.); but if BD, CE are to lie in opposite directions, (fig. 5.) find F the vertex of the segment BAC, and in either case join FG, which produce to meet the circle in H the point to be found; that is, if any circle be described through H and A to meet the given lines in D and E, BD is to CE as α to β . Join HB, HC, HD, HE. The triangles BDH, CEH are similar, for the angle BDH is equal to CEH, and because the angle BHC is equal to DHE, therefore BHD is equal to CHE; hence BD is to CE as BH to HC, that is, (because HG bisects the angle BHC), as BG to GC, or as α to β .

It is evident that the point H may be also found, by taking any segments BD, CE, in the given ratio of α to β , and describing a circle through the points D, A, E, to meet the circle BAFC in H the point required. If the given lines be parallel, and the points B, C, also the ratio of BD to CE, (fig. 6.) given as before, the indeterminate circle will be changed into a straight line passing through a given point H, which will be without the given lines, or between them, according as BD, CE, are to lie in the same, or in contrary, directions with AB, AC.

PROP. II. PORISM, Fig. 7, Pl. I.

Let AF, AG, be two straight lines given by position, a point H may be found, such, that any circle whatsoever described through it, and A the intersection of the given lines, to meet them in D and E, shall cut off from them segments AD, AE, whose sum shall be a given line M.

Suppose the porism to be true, and that the point is found, and circle described as above, let given points B, C be so taken, that BA and AC may be together equal to DA and AE , that is, by hypothesis, to the given line M , then BD will be equal to CE . If a circle be described through the given points A, B, C , by hypothesis it will meet the circle passing through A, D, E , in H , the point which may be found. Join BH, CH, DH, EH . The angle BHC is equal to DHE , each being the supplement of BAC , therefore BHD is equal to CHE ; now, HDB is equal to HEC , and BD is equal to CE , therefore the triangle HBD is equal to HCE , and BH is equal to CH , also DH to EH ; hence the angle BAH is equal to CAH , and H is in a straight line bisecting the angle FAG , but it is also in the given circle BAC ; therefore the point H is given, as was required.

Hence this construction: Take B and C , two given points, so that BA and AC may be together equal to M , and through A, B, C , describe a circle. Draw AK bisecting the given angle FAG , and meeting the circle ABC in H the point required, that is, if any circle be described through H and A , to meet the given lines in D and E , the sum of DA and AE shall be equal to the sum of BA and AC , that is, by construction, to the given line M . The synthetical demonstration follows readily from the preceding analysis.

PROP. III. PORISM, Fig. 8, Pl. II.

Let AF, AG , be two straight lines given by position, a point H may be found, such, that if any circle be described through it, and A the intersection of the given lines, to meet them in D and E , the difference between AD and AE shall be equal to a given line N .

The analysis of this proposition will differ in nothing material from the last, and the point required may be found thus: Take B and C , two given points, so that the difference between BA

and AC may be equal to N. Through the points A, B, C, describe a circle. Draw AK bisecting the angle contained by FA one of the given lines, and AL the other line produced at their intersection, and AK will meet the circle ABC in H, the point which may be found; that is, if any circle be described through H and A, to meet the given lines in D, E, the difference between AD and AE is equal to N the given line.

Join BH, CH, DH, EH. The triangles HCE, HBD are equal to one another in every respect, for if BC be joined, the angle HBC is equal to HAL, that is, by construction, to HAB, therefore HB is equal to HC; in the same way it appears that HD is equal to HE; now, the angle DHE is, equal to DAE, that is to BHC, therefore BHD is equal to CHE, hence BD is equal to CE, and the difference between DA and AE is the same with the difference between BA, AC, which by construction is equal to the given line N.

These two last propositions may be considered as particular cases of the following proposition.

PROP. IV. PORISM, Fig. 4, 5. Pl. I.

Two straight lines AB, AC being given by position, and two lines P, Q being given in magnitude, a point H may be found, (fig. 5.) such, that any circle described through it and A the intersection of the given lines, to meet them in D, E, shall cut off from them segments AD, AE, so that $P \times AD + Q \times AE$, shall be equal to a given space. Also, the same things being supposed, a point H may be found, (fig. 4.) so that $P \times AD - Q \times AE$, shall be equal to a given space.

Let given points B, C, be taken in either case agreeing with the hypothesis of the proposition, or so that $P \times AB + Q \times AC$, (fig. 5.) may be equal to $P \times AD + Q \times AE$, and so that

$P \times AB - Q \times AC$ may be equal to $P \times AD - Q \times AE$, (fig. 4.) then, in both cases, $P \times BD$ will be equal to $Q \times CE$; therefore BD is to CE as Q to P , that is, in a given ratio, and the points B, C being given, the point H may be found (Prop. 1.).

Construction. Let given points B, C , be taken as above directed, and if $P \times AD + Q \times AE$ is to be a given space, (fig. 5.) find a point H , (Prop. 1.) so that any circle described through A and H may meet the given lines in D, E , so that BD, CE may lie in contrary directions to AB, AC , and have to each other the given ratio of Q to P , then $P \times BD$ will be equal to $Q \times CE$, and adding the common space $P \times AB + Q \times AE$ to each, we get $P \times AD + Q \times AE$, equal to $P \times AB + Q \times AC$, that is, to the given space, as was required.

But if $P \times AD - Q \times AE$ is to be a given space, (fig. 4.) find H , (Prop. 1.) so that any circle passing through H, A , may cut off segments BD, CE , in the given ratio of Q to P , and lying towards the same parts with AB, AC , then $P \times BD$ is equal to $Q \times CE$, and $P \times AD - Q \times AE$, will be equal to $P \times AB - Q \times AC$, that is, by construction, to the given space.

LEMMA, Fig. 9. Pl. II.

If circles be described through A and C any two angles of a triangle ABC , to meet each other at D a point in AC , and the remaining lines AB, BC , in E and F ; their other intersection H , the remaining angle B , and the points E, F , are in the circumference of a circle.

Join DH, EH, FH . The angle AEH is equal to ADH or CFH , that is, BEH is equal to BFH , hence the points H, B, D, F are in a circle. Q. E. D.

PROP. V. PORISM, Fig. 10. Pl. II

Let AB, AC, BC , be three straight lines given by position, a point H may be found, such, that if any circle be described through H and B the intersection of any two of the given lines, to meet them in D and F , and if DF be joined meeting the remaining line at E , the line DF shall be divided at E , into segments having to each other a given ratio.

Suppose that the point H is found. Join HA, HB, HC ; join also HD, HE, HF . Since, by hypothesis, a circle may pass through the point which is to be found, the intersection of any two of the given lines, and the points where DF meets these lines, therefore the points H, A, D, E are in a circle, and the angle HEF is equal to HAD or HAB ; now the points H, B, D, F are supposed to be in a circle; since therefore in the triangle ABC , circles pass through two of its angles A, B , and meet each other at D , a point in AB , (Lemma.) the points H, C, E, F are also in a circle; therefore the angle HCF is equal to HEF , that is, (as has been shewn), to HAB ; hence the point H , which may be found, is in a circle passing through the points A, B, C , whatever be the given ratio of DE to EF . Let this circle be described.

Because the points H, A, D, E are in a circle, the angle HAC is equal to HDE , and because H, C, E, F are in a circle, the angle HFE is equal to HCA ; therefore the triangles AHC, DHF are similar. In the same manner it appears, that AHB is similar to EHF , and CHB to EHD .

Let AC be divided at K , so that AK may be to KC , in the given ratio of DE to EF , the point K will thus be given. Join HK meeting the circle in G . The triangles AHC, DHF being similar, and having AC, DF , similarly divided at K, E , the tri-

angles AHK, KHC will therefore be similar to DHE, EHF, which have been proved similar to BHC, AHB; therefore the angle AHB is equal to CHK or CHG, and the arch AB is equal to CG, hence G is a given point, and K being given, the line GH will be given by position; therefore the point H is given, which was to be found.

Construction. Describe a circle through the points A, B, C, let AB, BC, be the lines upon which D and F, the extremities of the indeterminate line, are to be placed, and let AC be the line which is to meet it in E, so that DE may be to EF, in the given ratio of de to ef . Find K, so that AK may be to KC as de to ef , draw BG parallel to AC, meeting the circle in G, join GK meeting the circle in H, the point which may be found; that is, if any circle be described through H, and B the intersection of any two of the given lines, to meet them in D and F, and if DF be joined, meeting the remaining line at E, the line DF shall be divided at E, similarly to the given line de .

Let AH, BH, CH be joined, also DH, EH, FH. The angle HDF or HDE is equal to HBF, that is, to HAE, the points H, A, D, E are therefore in a circle, now the points H, B, D, F are in a circle, therefore (Lemma.) the points H, C, E, F are also in a circle. The angle HDE is equal to HBC, that is, to HAK, and since HEF is equal to HCF, therefore HED is equal to HCB, that is, to HGB or HKA; hence the triangles HDE, HAK are similar, and since HFE is equal to HCK, the triangles HEF, HKC are also similar; therefore DE is to EF as AK to KC, that is, as de to ef .

Cor. 1. The lines DH, EH, FH contain given angles, and have to each other the given ratios of AH, KH, CH.

Cor. 2. The line DF cuts off segments DA, EK, FC from the given lines, adjacent to given points in them, and having to each other the given ratios of HA, HK, HC. For the angles HDB, HEK, HFC are equal among themselves, and since BCH

or BGH, that is, AKH, is the supplement of each of the angles HCE, HAD, HKE, the angles HAD, HKE, HCF are equal among themselves, therefore the triangles HAD, HKE, HCF are similar, and AD, KE, CF are proportional to the given lines AH, KH, CH.

PROP. VI. PORISM, Fig. 11. Pl. II.

Let AB, AC, BE, DE be four straight lines given by position; a point P may be found, such, that if any circle be described through it and B, any of the six intersections of the given lines, to meet the lines through whose intersection it passes in G and L, and if GL be joined, meeting the remaining lines in H and K, the segments GH, HK, KL have given ratios to one another, which ratios are to be found.

Because, by hypothesis, the points P, A, G, H are in a circle, and also the points P, F, H, K, it will appear, as in the analysis of the last proposition, that P is in a circle described about the triangle ADF; in the same way it will be found, that P must be in circles described about each of the triangles ABC, DBE, FCE. Therefore, that the proposition may be universally true, these four circles must intersect one another at the same point.

About any two of these triangles, as ABC, DBE, let circles be described, the point P must be at their intersection.

Because ADF is a triangle, and through two of its angles A, D, circles are described, meeting each other at B, a point in AD, therefore (Lemma.) P, their other intersection, and the points F, C, E, are in a circle; and because FCE is a triangle, and circles pass through C, E, two of its angles, and meet each other at B, a point in CE, therefore (Lemma.) the points P, A, D, F are in a circle. Thus it appears, that circles described

about each of the four triangles ADF, ABC, DBE, CFE, pass through the same point P as was to be investigated. It remains to inquire, whether the ratios of GH, HK, KL to one another be given. Join PB, PC, PE, also PG, PH, PK, PL. The angle GPH is equal to GAH, that is, to BPC, and PGH is equal to PBC, therefore the triangles BPC, GPH are similar, and the angle PHK is equal to PCE; but HPK is equal to HFK, that is, to CFE or CPE, hence the triangles HPK, CPE are similar, and PKL is equal to PEL. Now, if PN be drawn, so that the angle BPN may be equal to GPL, that is, to the given angle GBL, it is evident that the point N is given, and will be in a circle passing through P, and touching AG at B; the angles NPE, LPK will thus be equal, and the triangles NPE, LPK similar. Since, therefore, the triangles BPC, CPE, EPN are similar to GPH, HPK, KPL, it follows, that BN, GL are similarly divided by the given lines CH, EK, therefore the ratios of GH, HK, KL are the same with the given ratios of BC, CE, EN.

Construction. About ABC, DBE any two of the four triangles formed by the given lines, let circles be described, they will meet each other at P, the point which is to be found.

Through P and B, the intersection of any two of the given lines, let a circle be described to touch one of them at B, and cut the other at N, the line BN will be given, and the ratios of GH, HK, KL, the same with the given ratios of BC, CE, EN to one another.

The synthetical demonstration follows readily from the analysis, and for the sake of brevity is here omitted.

Cor. 1. The lines PG, PH, PK, PL, contain given angles, and have to each other the given ratios of PB, PC, PE, PN.

Cor. 2. The line GL cuts off from the given lines, segments BG, CH, EK, NL, adjacent to given points, and having to each other the given ratios of PB, PC, PE, PN. For the points P, A, G, H, being in a circle, the angle PGB is equal to PHC; and

since P, F, H, K , are in a circle, the angle PHC is equal to PKE , which in like manner will be found equal to PLN . Now, the angles PBA, PCF, PEF, PNB are equal among themselves, therefore their supplements PBG, PCH, PEK, PNL are equal, and the triangles PBG, PCH, PEK, PNL are similar, therefore, BG, CH, EK, NL are proportional to the given lines BP, CP, EP, NP .

PROP. VII. THEOREM, Fig. 12. Pl. III.

Let $PGAB, PFAC, PEAD, \&c.$ be any number of given circles, each of which passes through the same two points A, P ; from A , either of these points let a straight line, given by position, be drawn, meeting the circles at $B, C, D, \&c.$ and another meeting them at $E, F, G, \&c.$ Let straight lines $GB, FC, ED, \&c.$ be drawn, joining these points, so as to form, with the lines passing through A , triangles $GAB, FAC, EAD, \&c.$ in each of the circles. If, through P , the common intersection of the circles, and Q , the intersection of any two of the lines, a circle be described to meet them in K and L , a line joining KL , and meeting the remaining lines, will be divided by them into segments $HK, KL, LM, MN, \&c.$ having to each other given ratios.

Let $Q, R, S, \&c.$ be the remaining intersections of $GB, FC, ED, \&c.$ Because GRE is a triangle, and circles $PGAB, PEAD$ pass through G, E , two of its angles, and meet at A , a point in GE , the points P, R, B, D , are in a circle, (Lemma.) in the same way it appears, that circles may pass through P, S, C, D , and $P, Q, B, C, \&c.$ Because it is now proved, that in the triangle CDS , a circle may pass through P, C, B, Q , and another through P, D, B, R ; therefore the points P, S, R, Q , are in a circle. (Lemma.) Thus it may be shewn, that circles described about each of the triangles, formed by the intercepted segments

of the straight lines, will all pass through the same point P. From P draw straight lines to the points of intersection of one of the given lines, with all the others, as PA, PB, PC, PD, &c. Join PH, PK, PL, PM, PN, &c.

Since P, Q, K, L, are in a circle, the angle BKP is equal to CLP; now, the angles PBG, PCF, are each equal to PAG; therefore the angles PBK, PCL, are equal, and the triangles PBK, PCL, similar; hence KP is to PL as BP to PC; now the angle KPL is equal to KQL, that is, to BPC; therefore the triangles KPL, BPC, are similar, and the angle PLM will be equal to PCD. But the points P, S, C, D, having been proved to lie in a circle, if PS be joined, the angle PCD will be equal to PSD, therefore PLM is equal to PSD or PSM, hence the points P, S, L, M are in a circle. In the same way it may be shewn, that P, G, H, K are in a circle, as also P, D, M, N, &c. and that the triangles PAH, PDM, &c. are each similar to PBK and PCL, and hence that PHK is similar to PAB, and PLM to PCD, &c. Through P describe a circle to touch AG at A, and meet AD in V, which will be a given point, since GA, AD, are given by position.

Join PV, the angle PVA is equal to PAE or PDS, that is, (P, D, M, N being in a circle) to PNM, and PDV is equal to PMN, the triangle PMN is therefore similar to PDV; and since the angle PVA is equal to PDS, also PNV to PMD, the triangles PDM, PVN, are similar. Thus it appears, that HN and AV are similarly divided by the lines BK, CL, DM, &c.; now, the points A, B, C, D, V, &c. are given; therefore the ratios of HK, KL, LM, MN, &c. to one another are given. Q. E. D.

Cor. 1. The lines PH, PK, PL, PM, PN, &c. contain given angles, and have to each other the given ratios of PA, PB, PC, PD, PV, &c.

Cor. 2. The line HN cuts off from the given lines, segments HA, KB, LC, DM, VN, &c. adjacent to given points, and hav-

ing also to one another the given ratios of PA PB, PC, PD, PV, &c.; for the triangles PAH, PBK, PCL, PDM, PVN, &c. have been proved equiangular; and therefore AH, BK, CL, DM, VN, &c. are proportional to PA, PB, PC, PD, PV, &c.*

PROP. VIII. PORISM, Fig. 13, Pl. II.

Let CA, CB, AB be three straight lines given by position, a point H may be found, such, that if through H, and B, C, any two of the intersections of these lines, there be described circles HBEF, HCDE, to meet each other at E, a point in BC, and the remaining lines at D and F. If DE, EF, DF be joined, the triangle DEF shall be similar to a given triangle *def*, and shall have its angles upon the given lines in a given order.

Because circles are described through C, B, and meeting each other at E, a point in CB, therefore their other intersection H, the remaining angle A, and the points D, F, are in a circle. (Lemma.) Let a circle be described through H, C, A, to meet CB in G, and another through H, B, G, to meet AB in K. Join HA, HG, HK, also HD, HE, HF. The angles ADH, GEH, KFH, are equal to one another, and the angles CAH,

* It may be proper to remark here, that, in the preceding propositions, the straight lines given by position, as well as the indeterminate straight line, which is cut by them into segments, having to each other given ratios, and which also cuts off from them segments adjacent to given points, and having to each other given ratios, are tangents to a parabola, of which the point that is required to be found is the focus. This consideration suggests some curious propositions, relating to tangents to the parabola. Some of them have been observed by Dr. Halley, in his translation of the *Sectio Rationis* of Apollonius.

One very obvious application of the propositions above hinted at, is to describe parabolas that shall pass through given points, and touch straight lines given by position.

$\angle CGH$, $\angle BKH$ are equal, therefore $\angle HAD$, $\angle HGE$, $\angle HKF$ are equal, and the triangles HAD , HGE , HKF are similar; therefore DH is to HE as AH to HG , and EH is to HF as GH to HK ; now, the angles DHE , EHF are equal to $\angle DCE$, $\angle EBF$, that is, to $\angle AHG$, $\angle GHK$; hence if AG , GK be joined, the quadrilateral $HDEF$ is similar to $HAGK$, and the triangle DEF is similar to AGK ; now, the angles EDF and DEF are given by hypothesis, therefore $\angle GAK$ and $\angle AGK$ are given; but A is a given point, and AK is given by position, therefore AG and the point G are given; therefore GK and the point K are also given, and H , the intersection of the given circles GAC , GBK , will be given, which was to be found.

Construction. Take a given point, which, to render the construction more simple, may be at A , one of the intersections of the given lines. Let AG , GK be so drawn as to form a triangle AGK , similar to the given triangle $d e f$, and having its angles placed upon the given lines, in the given order. Through A , G , any two of its angles, and C , the intersection of the lines upon which they are placed, describe a circle; through G , K , and B , the intersection of CG , AK , let another circle be described, meeting the former in H , the point to be found, which will also be in a circle passing through K , and touching CA at A .

The demonstration follows easily from the preceding analysis.

Cor. 1. The lines HD , HE , HF contain given angles, and have to each other the same ratios, with the given lines HA , HG , HK .

Cor. 2. The lines AD , GE , KF have also to each other the given ratios of HA , HG , HK .

PROP. IX. THEOREM, Fig. 14. Pl. III.

Let Ea , Eb , Fc , Gd , &c. be any number of straight lines given by position. Let P be a given point. Through P , and E ,

the intersection of any two of the given lines, let a circle be described to meet them in A and B ; through P , B , and H , the intersection of Bb , with one of the remaining lines, let a circle be described to meet that line in C . Through P , C , and K , the intersection of Cc , with one of the remaining lines, let a circle be described to meet that line in D , and so on if there be more lines. Join AB , BC , CD , &c. DA . The rectilineal figure $ABCD$, &c. is given in species.

Take a , a given point in EA , through P , E , a , describe a circle to meet EB in b , through P , H , b , describe a circle to meet HC in c , through P , K , c , describe a circle to meet KD in d , and so on if there be more lines. Join Pa , PA , also PB , Pb , PC , Pc , PD , Pd , &c. Because the points P , E , A , B , are in a circle, the angle PAa is equal to PbB ; now PaA is equal to PbB ; for PaE is equal to PbE , the triangles PaA , PbB are therefore similar. In the same manner it may be shewn, that PbB is similar to PcC , and that again to PdD , &c. Therefore PA is to PB as Pa to Pb , and PB to PC as Pb to Pc , and PC to PD as Pc to Pd , &c.; now the angles APB , BPC , CPD , &c. are equal to AEB , BHC , CKD , &c. that is, to aPb , bPc , cPd , &c. therefore if ab , bc , cd , &c. Ad be joined, the rectilineal figure $PABCD$, &c. is similar to $Pabcd$, &c.; and leaving out the similar triangles PAD , Pad , the rectilineal figure $ABCD$, &c. is similar to $abcd$, &c. Now the points P , E , a , being given, the circle passing through them is given; therefore b is a given point; in like manner c , d , &c. are given points; therefore the figure $abcd$, &c. is given; therefore $ABCD$, &c. to which it is similar, is given in species, Q. E. D.

Cor. 1. The lines PA , PB , PC , PD , &c. contain given angles, and have to each other the given ratios of Pa , Pb , Pc , Pd , &c.

Cor. 2. The segments Aa , Bb , Cc , Dd , &c. of the given lines, adjacent to the given points a , b , c , d , &c. have also to each other the given ratios of Pa , Pb , Pc , Pd , &c.

Cor. 3. If there be any number of straight lines given by position, there may be innumerable rectilineal figures similar to one another, and having their angles upon the straight lines given by position.

PROP. X. PORISM, Fig. 15, Pl. III.

Let A and B be two given points in the circumference of a given circle. Let C be a given point in KC , a straight line given by position. There may be found a straight line KD given by position, and also a given point D in that line, such, that if AE , BE be inflected to any point in the circumference of the given circle, they shall cut off from KC , KD , segments FC , GD , adjacent to the given points, and having to each other the given ratio of α to β .

Suppose the line KD , and the point D to be found. If AH , BH be inflected to the circle, so that AH may pass through C , then BH must pass through D , the point which may be found, otherwise the proposition would not be universally true. Now, C being given, the point H , and the line BH , will be given by position. Let AL be drawn parallel to KC , then BL must be parallel to KD , the line to be found; hence it appears, that the angle GKF is equal to ALB , that is, to GEF ; therefore the points E , K , G , F are in a circle, and the angle DGB is equal to CFA ; now DBG is equal to CAF ; therefore the triangles DBG , CAF are equiangular, and AC is to BD as CF to DG , that is, by hypothesis, as α to β ; now AC is given, and BH is given by position, therefore the point D is given, but BDG is equal to the given angle ACF , therefore DG is given by position.

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